INTERNATIONAL BACCALAUREATE
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MARKSCHEME

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FURTHER MATHEMATICS

Standard Level

Paper 2

15 pages

1. (i) (a) $\left(e^{x^n}\right)' = nx^{n-1}e^{x^n}$ ***(A1)***

$f(x) = e^{-x^3} - e^{-x^2} \Rightarrow f'(x) = -3x^2e^{-x^3} + 2xe^{-x^2}$ ***(M1)(AG)***

[2 marks]

(b) The function will have a maximum where $f'(x) = 0$. From the graph of the function this happens at a positive point near zero,

$f'(x) = 0 \Rightarrow 2xe^{-x^2} - 3x^2e^{-x^3} = 0$

$\Rightarrow x\left(2e^{-x^2} - 3xe^{-x^3}\right) = 0$ ***(C1)***

Since x must be different from zero, ***(R1)***

$\Rightarrow 2e^{-x^2} - 3xe^{-x^3} = 0 \Rightarrow 3xe^{-x^3} = 2e^{-x^2}$ ***(M1)***

$\Rightarrow x = \frac{2e^{-x^2}}{3e^{-x^3}} = \frac{2}{3}e^{x^3-x^2}$

So, we set up the iteration as

$x_{n-1} = \frac{2}{3}e^{x_n^3 - x_n^2}$ ***(A1)***

Start with $x_0 = 0.5$,

$x_1 = 0.588\ldots$

$x_2 = 0.578\ldots$

With $x_4 = 0.579$, hence the point is (0.579, 0.108) ***(A1)***

[5 marks]

(c) The two points that bound the region are 0 and 1 since $f(x) = 0$ if and only if $e^{-x^2} = e^{-x^3}$ which can only be true when $x^3 = x^2$, ***(M1)***

i.e. $x = 0$ or $x = 1$ ***(A1)***

$T = \frac{1-0}{2\times5}\left[f(0) + 2\left(\left(f(0.2) + f(0.4) + f(0.6) + f(0.8)\right) + f(1)\right]$ ***(C1)***

$T = 0.1\left[0 + 2\left(e^{-0.2^3} - e^{-0.2^2} + e^{-0.4^3} - e^{-0.4^2} + e^{-0.6^3} - e^{-0.6^2} + e^{-0.8^3} - e^{-0.8^2}\right) + 0\right]$

$T = 0.0594$ ***(A1)***

OR

$T = 0.0594$ ***(G2)***

[4 marks]

continued...

Question 1 continued

(ii) Use the absolute ratio test to get

$$\lim_{n\to\infty}\left(\left(\frac{|x-3|^{n+1}}{2^{n+1}\sqrt{n+1}}\right)\times\left(\frac{2^n\sqrt{n}}{|x-3|^n}\right)\right)=\lim_{n\to\infty}\left(\left(\frac{|x-3|}{2}\right)\times\left(\sqrt{\frac{n}{n+1}}\right)\right)$$ ***(M1)(R1)***

$$=\frac{|x-3|}{2}$$ ***(A1)***

Hence the series converges when

$$\frac{|x-3|}{2}<1\Leftrightarrow|x-3|<2\Leftrightarrow 1<x<5$$ ***(M1)***

so the series converges in the open interval $]1, 5[$. ***(C1)***

Now we must check the endpoints:

For $x=1$ the series becomes $\sum_{n=1}^{\infty}\frac{(1-3)^n}{2^n\sqrt{n}}=\sum_{n=1}^{\infty}\frac{(-2)^n}{2^n\sqrt{n}}=\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ which converges by the alternating series test since the terms decrease to zero in magnitude. ***(M1)(A1)***

For $x=5$, the series becomes $\sum_{n=1}^{\infty}\frac{(5-3)^n}{2^n\sqrt{n}}=\sum_{n=1}^{\infty}\frac{(2)^n}{2^n\sqrt{n}}=\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ which diverges since it is a *p*-series with $p=\frac{1}{2}<1$. ***(R1)***

So the interval of convergence is $[1, 5[$. ***(A1)***

[9 marks]

Total [20 marks]

2. (a) Closure $\begin{pmatrix} x & x \\ x & x \end{pmatrix}\begin{pmatrix} y & y \\ y & y \end{pmatrix} = \begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix} \in G$ ***(M1)(C1)***

Associativity is assumed under matrix multiplication. ***(C1)***

$\begin{pmatrix} x & x \\ x & x \end{pmatrix}\begin{pmatrix} a & a \\ a & a \end{pmatrix} = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$ ***(M1)***

$\Rightarrow 2ax = x$ ***(M1)***

$\Rightarrow a = \frac{1}{2}$

$\Rightarrow \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ is the identity element ***(A1)***

The inverse $\begin{pmatrix} x & x \\ x & x \end{pmatrix}\begin{pmatrix} b & b \\ b & b \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ ***(M1)***

$\Rightarrow 2bx = \frac{1}{2} \Rightarrow b = \frac{1}{4x}$ ***(M1)***

$\Rightarrow \begin{pmatrix} \frac{1}{4x} & \frac{1}{4x} \\ \frac{1}{4x} & \frac{1}{4x} \end{pmatrix}$ is the inverse. ***(A1)***

$\begin{pmatrix} x & x \\ x & x \end{pmatrix}\begin{pmatrix} y & y \\ y & y \end{pmatrix} = \begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix} = \begin{pmatrix} y & y \\ y & y \end{pmatrix}\begin{pmatrix} x & x \\ x & x \end{pmatrix}$ ***(R1)***

Therefore, the set is an Abelian group. ***(AG)***

[10 marks]

(b) This we prove by mathematical induction

For $n = 1, (a^{-1}ba)^1 = a^{-1}ba$ ***(C1)***

Assume relation true for $n = k$, i.e., $(a^{-1}ba)^k = a^{-1}b^k a$ ***(C1)***

Prove true for $n = k + 1$:

$(a^{-1}ba)^{k+1} = (a^{-1}ba)^k (a^{-1}ba)$

$= (a^{-1}b^k a)(a^{-1}ba)$ by assumption ***(M1)***

$= (a^{-1}b^k)(aa^{-1})(ba)$ associative property

$= (a^{-1}b^k)e(ba)$ identity property

$= a^{-1}(b^k b)a$ associative property ***(M1)***

$= a^{-1}b^{k+1}a$ ***(A1)***

Hence the relation is true by mathematical induction.

[5 marks]

(c) Let $x \neq y \in A$, then $g \circ f(x) = x$ and $g \circ f(y) = y$ ***(M1)(R1)***

suppose $f(x) = f(y)$, then $g\left(f(x)\right) = g\left(f(y)\right)$ since g is a function ***(R1)***

hence $x = y$ by definition, which is a contradiction, ***(M1)***

therefore $f(x) \neq f(y)$, and $f(x)$ is injective. ***(R1)***

[5 marks]

Total [20 marks]

3. (i) (a) Since $\deg_{G'}(v) = (n-1) - \deg_G(v)$, ***(R1)***

and since $n = 14$, so the degree sequence G' is created by using: $13 - \deg(v)$ for each vertex in the opposite order $13 - 5$, *etc*... ***(M1)(A1)***

8, 8, 9, 9, 9, 10, 10, 10, 11, 11, 11, 12, 12, 12. ***(A1)***

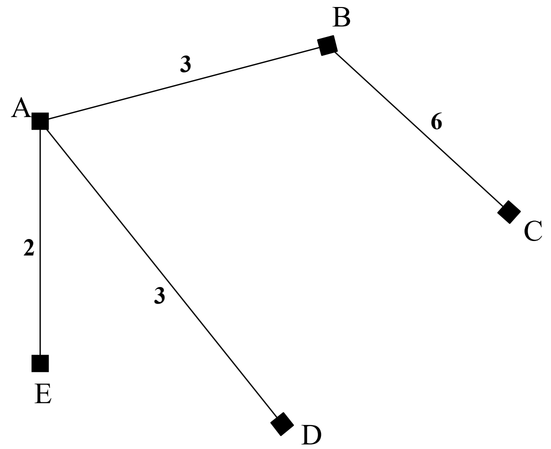
[4 marks]

(b) Kruskal algorithm

The algorithm is:

"For a graph with n nodes keep adding the shortest (least cost) link – *avoiding* the creation of *circuits* – until $(n-1)$ links have been added." ***(R1)***

Note here that the Kruskal algorithm only applies to graphs in which all the links are *undirected*. For the graph shown above, applying Kruskal algorithm and starting with the shortest (least cost) link, we have:



(A1)

Link	Cost	Decision
A–E	2	add to tree
A–B	3	add to tree
A–D	3	add to tree
E–D	4	reject as forms circuit 1–5 –4 –1
E –B	5	reject as forms circuit 1–5 –2 –1
B–C	6	add to tree

Stop as 4 links have been added and these are all we need ***(M2)***

So, Minimum cost is 14. ***(A1)***

[5 marks]

Question 3 continued

(ii) (a) Either $3|a$ or $3\nmid a$ (for either *a* or *b*). ***(M1)***

In the second case either $a \equiv 1 \bmod 3$ or $a \equiv 2 \bmod 3$ (and the same for *b*), and in both cases it follows that $a^2 \equiv 1 \bmod 3$ and $b^2 \equiv 1 \bmod 3$, hence $a^2 + b^2 \equiv 1 \bmod 3$ (when one of them is divisible by 3) or $a^2 + b^2 \equiv 2 \bmod 3$ which contradicts the hypothesis. Therefore the result follows. ***(M1)(R1)(C1)(R1)***

[5 marks]

(b) If $p|a$ then $p|a^2$ and since $p|a^2 + b^2$ then $p|(a^2 + b^2 - a^2)$ ***(R2)***

So, $p|b^2$. Since *p* is prime, *p* must divide *b*. ***(R1)***

[3 marks]

(c) If $(a, b) = 1, \Rightarrow$ there are two integers *s* and *r* such that: $ra + sb = 1$, ***(M1)***

If $(a, c) = d, \Rightarrow$ there are two integers *p* and *q* such that: $pa + qc = d$, ***(M1)***

Then $pa + qc(ra + sb) = d$, and hence $(p + qcr)a + (qs)(bc) = d$ and hence the result follows. ***(M1)(R1)***

[4 marks]

Total [21 marks]

4. (a) The mean grade is $\frac{\sum xf(x)}{\sum f(x)} = \frac{1\times 10 + \ldots 7\times 6}{158} = \frac{581}{158}$ ***(M1)***

$= 3.68$ ***(A1)***

[2 marks]

(b) (i)

Grade	1	2	3	4	5	6	7
Expected number of candidates	2.6	**13.2**	37.1	52.3	**37.1**	**13.2**	2.6

(A2)

(ii) To check whether the data is normal or not, we must run a χ^2 goodness of fit test. ***(C1)***

H_0 : Data is normal with mean 4 and standard deviation 1.17.

H_1 : Data is not normal with the same mean and standard deviation. ***(C1)***

The first two and last two classes must be combined since the expected number is less than 5. ***(C1)***

$$\chi^2_{calc} = \frac{(35-15.8)^2}{15.8} + \frac{(38-37.1)^2}{37.1} + \frac{(42-52.3)^2}{52.3} + \frac{(25-37.1)^2}{37.1} + \frac{(18-15.8)^2}{15.8}$$

$= 29.6$ ***(A1)***

The number of degrees of freedom is then $n-1 = 5-1 = 4$, hence the critical number is $\chi^2_c = 9.49$. ***(C1)***

Since $\chi^2_{calc} = 29.6 > \chi^2_c = 9.49$ we reject H_0 and conclude that the data is not normal with a mean of 4 and standard deviation of 1.17. ***(R1)***

[8 marks]

(c) To test the hypothesis, we do a Z-test:

$H_0 : \mu = 4$

$H_1 : \mu < 4$ ***(C1)***

This is a lower tail test, with $z_c = -1.645$

$z_t = \frac{3.677-4}{\frac{1.17}{\sqrt{158}}} = -3.47$ which lies in the rejection region. ***(M1)(A1)***

We reject H_0 and conclude that there is enough evidence to say that *Utopia's* performance is below that of the population. ***(R1)***

(Students may use a *p*-value of 0.0003 and draw the same conclusion. Accept the argument)

[4 marks]

continued...

Question 4 continued

(d) H_0 : Mathematics and Physics grades are independent.

H_1 : Mathematics and Physics grades are dependent.

This is a χ^2 contingency table analysis of independence with $(4-1)(4-1)=9$ degrees of freedom.
The expected frequencies are calculated by multiplying the row total by the column total and dividing by the number of observations in the sample. The expected matrix is shown below. ***(R1)(C1)***

	7	6	5	4 & below	Total
7	40	50	56	20	166
	34.72	**48.48**	**59.46**	**23.34**	
6	42	65	90	28	225
	47.06	**65.72**	**80.59**	**31.64**	
5	32	48	52	24	156
	32.63	**45.56**	**55.88**	**21.94**	
4 & below	60	80	100	45	285
	59.60	**83.24**	**102.08**	**40.08**	
Totals	174	243	298	117	832

$$\chi^2_{calc} = \sum_{\text{over all cells}} \frac{(f_o - f_e)^2}{f_e} = 0.804 + 0.047 + 0.201 + 0.479 + 0.543 + 0.008$$

$$+1.099 + 0.419 + 0.012 + 0.130 + 0.269 + 0.194$$ ***(M1)***

$$+0.003 + 0.126 + 0.042 + 0.604$$

$$= 4.981$$ ***(A1)***

$\chi^2_{calc} = 4.981 < \chi^2_c = 16.919$, we do not have enough evidence to reject H_0, ***(R1)***

i.e. we do not have enough evidence to conclude that there is any statistical dependence between the students' grades in physics and mathematics. ***(C1)***

[6 marks]

Total [20 marks]

5. (i) (a) In triangle ABC, [AF], [CE], and [BD] are concurrent at G. Hence, by Ceva's theorem ***(R1)***

$$\frac{CD}{DA}\times\frac{AE}{EB}\times\frac{BF}{FC}=1\Rightarrow\frac{3}{2}\times\frac{3}{2}\times\frac{BF}{FC}=1$$ ***(M1)***

$$\Rightarrow\frac{BF}{FC}=\frac{4}{9}$$ ***(A1)***

Also, [DH] intersects the three sides of this triangle, so, by Menelaus' theorem

$$\frac{CD}{DA}\times\frac{AE}{EB}\times\frac{BH}{HC}=1$$ ***(M1)***

$$\Rightarrow\frac{BH}{HC}=\frac{4}{9}$$ ***(A1)***

[5 marks]

(b) In triangle AFC, [DB] intersects the three sides, and by Menelaus' theorem

$$\frac{CD}{DA}\times\frac{AG}{GF}\times\frac{FB}{BC}=1$$ ***(M1)***

$$\Rightarrow\frac{3}{2}\times\frac{AG}{GF}\times\frac{FB}{BC}=1$$ ***(A1)***

However, $\Rightarrow\frac{BF}{FC}=\frac{4}{9}$ ***(M1)***

$$\Rightarrow\frac{BF}{BC}=\frac{4}{13},\ \frac{AG}{GF}=\frac{13}{6}$$ ***(A1)***

[4 marks]

continued...

Question 5 continued

(ii) $\frac{\text{PQ}}{\text{QR}} = \frac{3}{2} \Rightarrow \text{PQ} = \frac{3}{2}\text{QR}$ ***(C1)***

[QR] has a fixed length $\Rightarrow$ [PQ] also has a fixed length. ***(R1)***

The locus of P is a circle where centre is Q and radius $\frac{3}{2}$ QR ***(A3)***

Alternative solution if candidates took the ratio as PQ : PR

By Apollonius' theorem, the locus of points P is a circle. ***(R1)(A1)***
To find that circle, we consider two points, M and N on side [QR] such that
$\frac{\text{QM}}{\text{MR}} = -\frac{\text{QN}}{\text{NR}} = \frac{3}{2}$, the points are also fixed. ***(M1)***
By the bisector theorem, [PM] and [PN] are the internal and external bisectors of angle P. Hence angle MPN is a right angle, and the locus of points P is the circle with [MN] as diameter. ***(R1)(R1)***

[5 marks]

(iii) With A and A′ as vertices, we have

$\frac{\text{AF}}{\text{AE}} = e = \frac{c}{a} \Rightarrow$ ***(M1)***

$\text{AE} = \frac{a}{c} \times \text{AF} = \frac{a}{c}(c-a)$ ***(A1)***

$\frac{\text{MF}}{\text{MH}} = e \Rightarrow \text{MF} = e \times \text{MH}$ ***(M1)***

$\text{MF} + \text{MF}' = e(\text{MH} + \text{MH}') = e \times \text{HH}' = e(2 \times \text{AE} + \text{AA}')$ ***(M1)***

$= \frac{c}{a}\left(2 \times \frac{a}{c}(a-c) + 2a\right) = 2a - 2c + 2c$ ***(M1)***

$= 2a$ ***(AG)***

[5 marks]

Total [19 marks]